ISI UGB 2026 Solutions

Unofficial Solutions of UGB Paper

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These are unofficial solutions to ISI Entrance UGB Paper, 2026. The main motivation of writing these solutions is the absencse of any official solutions on the internet and to help the students with the technical details of proof-writing in a proof-based mathematical setting.

The tone of the solutions given here is very dry and very much to the point and I more or less expect a student to have certain mathematical maturity.

I understand that for some people these problems could be hard and quite unconventional. Keeping this in mind, I've not only put out the solution, but I've also written a blog post at

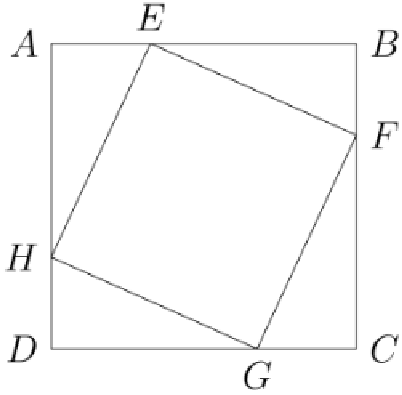
`https://satisfiedmagma.github.io/writings/posts/isi_2026_discussion/`

which doesn't have formal solutions, but rather an interesting commentary and the thought process to solve these problems. I agree that not everything can be motivated; a few techniques must be known beforehand but I've tried my best to put my thoughts on the blog as naturally and as clearly as possible.

Contents

0. Problems

1. The problem is in two parts.
   a) Let $a, b \in \mathbb{R}$ be such that the straight line $y = a + bx$ on the (x, y)-plane passes through (x_1, y_1) and (x_2, y_2). If x_1, y_1, x_2, y_2 are rational numbers with $x_1 \neq x_2$ and $y_1 \neq y_2$, show that a and b are rational numbers and $b \neq 0$.
   b) Let $\alpha, \beta, r \in \mathbb{R}$ be such that $r > 0$ and the circle $(x - \alpha)^2 + (y - \beta)^2 = r^2$ on the (x, y)-plane passes through (x_1, y_1), (x_2, y_2) and (x_3, y_3). Assume that x_1, x_2, x_3 are distinct rational numbers and y_1, y_2, y_3 are distinct rational numbers. Show that α and β are rational numbers.
2. Sketch with reasoning, the graph of $y = (\log_e x)^2 + (\log_e x)^{-2}$.
3. Suppose $n \geq 2$ is an integer. Define
$$S = \{(x_1, \ldots, x_n) : 0 < x_i < 1 \text{ for all } i = 1, \ldots, n\}$$
and a function $f\colon S \to \mathbb{R}$ by
$$f(x_1, \ldots, x_n) = \log_{x_1}((x_2)^2) + \cdots + \log_{x_{n-1}}((x_n)^n) + \log_{x_n} x_1.$$
Determine the minimum value of $f(x_1, \ldots, x_n)$ over S.
4. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that
$$f(x + y) + f(x - y) = 2(f(x) + f(y)) \text{ for all } x, y \in \mathbb{R}.$$
   a) If f' denotes the derivative of f, show that
$$f'(x + y) - f'(x - y) = 2f'(y) \text{ for all } x, y \in \mathbb{R}.$$
   b) Further assume that $f''(0) = 2$, where f'' is the second derivative of f. Determine f.
5. In the figure below, $ABCD$ is a square. The points E, F, G, H lie on AB, BC, CD, DA, respectively, such that $EFGH$ is a square. The perimeter of $ABCD$ exceeds the perimeter of $EFGH$ by 32 units. Determine the length of the radius of the in-circle of ΔEBF.



6. Let P be a polynomial of degree greater than or equal to one, with real coefficients, such that $P(x) \geq 0$ for all $x \in \mathbb{R}$.
   a) Suppose $x_0 \in \mathbb{R}$ is such that $P(x_0) = 0$. Show that there exist a positive integer n and a polynomial S with real coefficients such that $S(x_0) \neq 0$ and
   $$P(x) = (x - x_0)^{2n} S(x) \text{ for all } x \in \mathbb{R}.$$
   b) Hence or otherwise, show that there exist polynomials Q and R with real coefficients such that $R(x) > 0$ for all $x \in \mathbb{R}$ and
   $$P(x) = (Q(x))^2 R(x) \text{ for all } x \in \mathbb{R}.$$
7. Suppose P is a polynomial with integer coefficients.
   a) Show that for distinct integers m and n, $\frac{P(m)-P(n)}{m-n}$ is an integer.
   b) Hence or otherwise, prove that there do not exist distinct integers a, b, c such that
   $$P(a) = b, \quad P(b) = c, \quad \text{and} \quad P(c) = a.$$
8. Let k be a non-negative integer.
   a) Show that there exists a unique polynomial P_k of degree $k + 1$ with real coefficients such that
   $$\sum_{i=1}^{n} i^k = P_k(n) \text{ for all } n \geq 1.$$
   b) Determine the coefficient of x^{k+1} in $P_k(x)$.
   c) Hence or otherwise, determine the coefficient of x^k in $P_k(x)$.

1. UGB 2026/1

Problem 1 (UGB 2026/1)

The problem is in two parts.

a) Let $a, b \in \mathbb{R}$ be such that the straight line $y = a + bx$ on the (x, y)-plane passes through (x_1, y_1) and (x_2, y_2). If x_1, y_1, x_2, y_2 are rational numbers with $x_1 \neq x_2$ and $y_1 \neq y_2$, show that a and b are rational numbers and $b \neq 0$.

b) Let $\alpha, \beta, r \in \mathbb{R}$ be such that $r > 0$ and the circle $(x - \alpha)^2 + (y - \beta)^2 = r^2$ on the (x, y)-plane passes through (x_1, y_1), (x_2, y_2) and (x_3, y_3). Assume that x_1, x_2, x_3 are distinct rational numbers and y_1, y_2, y_3 are distinct rational numbers. Show that α and β are rational numbers.

Solution. We will split the solutions into two parts.

¶ Solution to part a) Since (x_1, y_1) and (x_2, y_2) lie on the line $y = a + bx$, upon substituting we get

$$y_1 = a + bx_1$$
$$y_2 = a + bx_2.$$

Subtracting the above two equations, we get

$$y_1 - y_2 = b(x_1 - x_2).$$

Since we are given that $x_1 \neq x_2$, therefore we can divide by $x_1 - x_2$ to get

$$b = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since $y_2 - y_1 \in \mathbb{Q}$ and $x_2 - x_1 \in \mathbb{Q}$ with $x_2 - x_1 \neq 0$(since we are given $x_1 \neq x_2$), we can conclude that $b \in \mathbb{Q}$ since quotient of two rational numbers is a rational. Finally, note that

$$y_1 - bx_1 = a$$

and since $y_1, x_1 \in \mathbb{Q}$ and we just proved $b \in \mathbb{Q}$, we get that $a = y_1 - bx_1 \in \mathbb{Q}$(since difference of two rationals is always rational) **finishing the proof to part a).** ■

¶ Solution to part b) To be added, for now visit the blog-post at

https://satisfiedmagma.github.io/writings/posts/isi_2026_discussion/#-problem-1

to see a sketch of the proof. **Thank You!**

■

2. UGB 2026/2

Problem 2 (UGB 2026/2)

Sketch, with reasoning, the graph of $y = (\log_e x)^2 + (\log_e x)^{-2}$.

Solution. For the entire solution, we will denote $\log_e(x)$ as $\ln(x)$. Define $\mathbb{R}_{>0}$ as set of positive reals. We will begin with the following simple observations.

- The **domain of expression** is $T := \mathbb{R}_{>0} \setminus \{1\}$. So, define a function $f\colon T \to \mathbb{R}$ such that
$$f(x) = (\ln x)^2 + \frac{1}{(\ln x)^2} \quad \text{for all } x \in T.$$
We need to graph f.

- f has **no real roots.** Indeed if $f(\alpha) = 0$ for some $\alpha \in T$, then
$$0 \le (\ln \alpha)^2 = \frac{-1}{(\ln \alpha)^2} < 0$$
which is absurd.

- f has **two vertical asymptotes**, one with the line $x = 0$ and one with the line $x = 1$(note that $0, 1 \notin T$). Indeed,
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left((\ln x)^2 + \frac{1}{(\ln x)^2}\right) = +\infty = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x).$$
Any other line is not an asymptote since f is a continuous function in its domain T.

We now analyze the derivative of f to find nature of monotonicity of f and find its local maxima/minima.

Claim — The function f is

- **strictly increasing** on $\left(\frac{1}{e}, 1\right)$ and (e, ∞);
- **strictly decreasing** on $\left(0, \frac{1}{e}\right)$ and $(1, e)$.

Moreover, f has a **local and a global minima** of 2 at $x = e, \frac{1}{e}$.

Proof. Since f is differentiable everywhere in its domain. Differentiating f with yields
$$f'(x) = \frac{2}{x}\left(\ln x - \frac{1}{(\ln x)^3}\right) \quad \text{for all } x \in T.$$

Observe that

$$f'(x) > 0 \iff \frac{2}{x}\left(\ln x - \frac{1}{(\ln x)^3}\right) > 0$$
$$\iff \frac{2}{x}\left(\frac{(\ln x)^4 - 1}{(\ln x)^3}\right) \cdot \left(\frac{x}{2}(\ln x)^4\right) > 0$$

$$\begin{aligned}
&\iff (\ln x)^3\left((\ln x)^4 - 1\right) > 0\\
&\iff \ln x \in (-\infty, 0) \cup (1, \infty)\\
&\iff x \in \left(\frac{1}{e}, 1\right) \cup (e, \infty).
\end{aligned}$$

Since the derivative is non-negative on $\left(\frac{1}{e}, 1\right)$ and (e, ∞), f must be strictly increasing on those intervals and strictly decreasing on intervals $\left(0, \frac{1}{e}\right)$ and $(1, e)$.

Further, its easy to see $f'(x) = 0 \iff x = e, \frac{1}{e}$. So, a local maxima/minima can only occur at $x = e$ or $x = \frac{1}{e}$. Since $f'(x) > 0$ for all $x > e$ and $f'(x) < 0$ for $x \in (1, e)$, by the **first-derivative test** of local minima at $x = e$, we get that $x = e$ must be the point of local-minima. Similarly, $x = 1/e$ is also a point of local minima. Also, $f(e) = f(1/e) = 2$.

By AM-GM Inequality,

$$f(x) = (\ln x)^2 + \frac{1}{(\ln x)^2} \geq 2\sqrt{(\ln x)^2 \cdot \frac{1}{(\ln x)^2}} = 2 \quad \text{for all } x \in T$$

which shows that 2 is indeed the **global minima** of f which **proves our claim.** □

¶ Curvature of f We wish to find when is f convex and when is f concave. We need a preparatory lemma to tackle this part.

Lemma

Consider $h(x) := x^5 - x^4 - x - 3$, then

$$h(x) > 0 \iff x > \beta$$

where $1 < \beta < 2$.

Proof. Note that

$$h'(x) = 5x^4 - 4x^3 - 1 = (x-1)(5x^3 + x^2 + x + 1)$$

For $x > 1$, $h'(x) > 0$. Thus h is strictly increasing over $(1, \infty)$. Also, $h(1) < 0$ and $h(2) < 0$. By *Intermediate Value Theorem*, there is some $\beta \in (1, 2)$ such that $h(\beta) = 0$. Since h is increasing on $(1, \infty)$, we get

$$x > \beta > 1 \implies h(x) > h(\beta) = 0 \quad \text{for all } x > \beta.$$

It is now sufficient to show that $h(x) < 0$ for all $x < \beta$. Define $r(x) := 5x^3 + x^2 + x + 1$. Then

$$r'(x) = 15x^2 + 2x + 1 \geq 2x^2 + 2x + 1 = (x+1)^2 + x^2 > 0$$

which shows that r is strictly increasing. Since r is a cubic polynomial, it must have at least root(which will be the only root since r is increasing) which we shall call α. Then we can write $r(x) = (x - \alpha)P(x)$ where P must be a strictly positive quadratic polynomial. Once again, since $r(-1) = -4 < 0$ and $r(0) = 1$, by *Intermediate Value Theorem*, $-1 < \alpha < 0$. Thus,

$$h'(x) = (x - \alpha)(x - 1)P(x).$$

This means h is increasing over $(-\infty, \alpha)$ and $(1, \infty)$ and decreasing over $(\alpha, 1)$. We now prove the following claim.

Claim (Computer Sorcery) — $h(x) < 0$ for all $x < 0$[a].

[a] Check out the blog here to find the reason for the name of this claim!

Proof. We will make two cases, whether $x \in (-\infty, -1)$ or $(-1, 0)$ or $x = -1$. The case $x = -1$ can be just checked by hand.

If $x < -1$, then $x^4 > 1 \iff x^4 - 1 > 0$. This gives $x(x^4 - 1) < 0$. Then

$$h(x) = x^5 - x^4 - x - 3 = x(x^4 - 1) - (x^4 + 3) < -(x^4 + 3) < 0.$$

This completes the proof to the first case. For the second case, assume $-1 < x < 0$. This gives $-2 < x - 1 < -1$ and $-3 < -x - 3 < -2$. Then

$$h(x) = x^5 - x^4 - x - 3 = x^4(x - 1) - x - 3 < -x^4 - x - 3 < -x^4 - 2 < 0$$

this concludes the proof. □

We now only need to show that $h(x) < 0$ for $0 < x < \beta$. We already know that h is decreasing over $(0, 1)$, which means

$$0 < x < 1 \implies 0 > -3 = h(0) > h(x) \quad \text{for all } x \in (0, 1).$$

Since h is increasing over $(1, \beta)$, we can similarly show that h is negative over $(1, \beta)$ as well. The graph of h is given below.

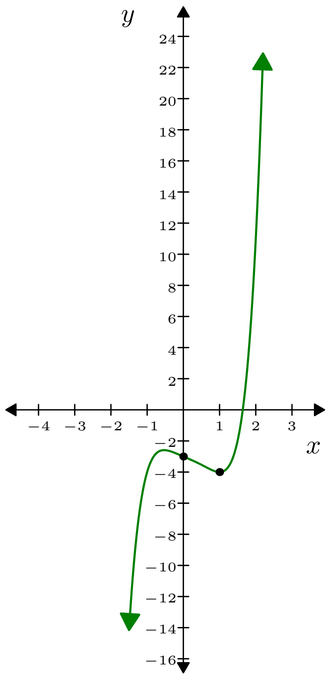


Figure 1: Rough Sketch of $h(x) = x^5 - x^4 - x - 3$

This indeed **completes the proof to this lemma.** □

To find the *curvature* of f, we need to compute the double derivative of f. Once again, f' is differentiable on T, thus we get

$$f''(x) = \frac{2}{x^2}\left(1 - \ln x + \frac{1}{(\ln x)^3} + \frac{3}{(\ln x)^4}\right)$$

for all $x \in T$. Note that

$$f''(x) > 0 \iff (\ln x)^5 - (\ln x)^4 - \ln x - 3 < 0 \iff h(\ln x) < 0, \quad \text{where } x > 0 \text{ and } x \neq 1$$

where we got the first iff condition by multiplying by $\frac{x^2}{2} \cdot (\ln x)^4 > 0$. Invoking our preparatory lemma, we get

$$f''(x) > 0 \iff h(\ln x) < 0 \iff \ln x \in (-\infty, \beta) \setminus \{0\}$$

which gives

$$f''(x) > 0 \iff x \in (\exp(-\infty), \exp(\beta)) \setminus \{e^0\} = (0, e^\beta) \setminus \{1\}.$$

Thus, by the **double-derivative test** of convexity, f is *convex* over $(0,1)$ and $(1, e^\beta)$. Similarly, f is *concave* over (e^β, ∞) where $1 < \beta < 2$. We now present two sketches of the graph, one is computer-generated and one is drawn by hand.

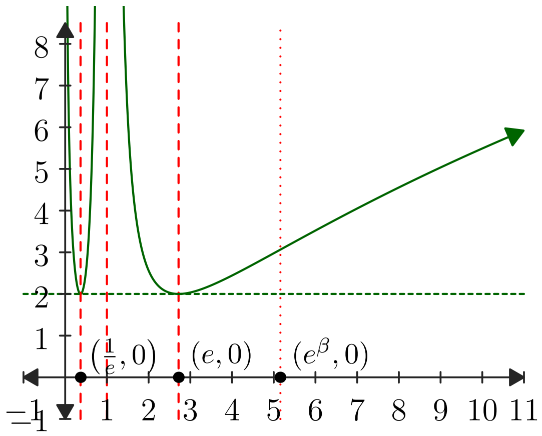


Figure 2: Computer-Generated Diagram of f

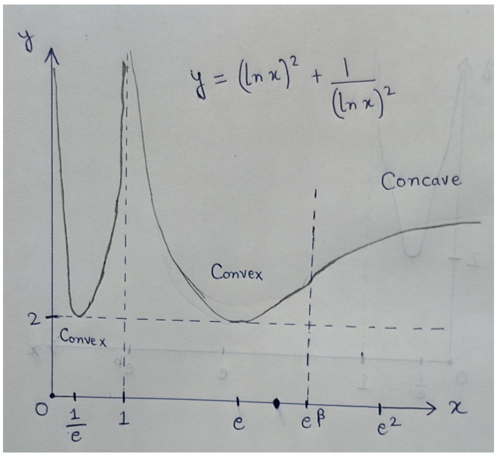


Figure 3: Handmade Graph of f

This indeed **completes the solution** to the problem. ■

3. UGB 2026/3

Problem 3 (UGB 2026/3)

Suppose $n \geq 2$ is an integer. Define

$$S = \{(x_1, \ldots, x_n) : 0 < x_i < 1 \text{ for all } i = 1, \ldots, n\}$$

and a function $f\colon S \to \mathbb{R}$ by

$$f(x_1, \ldots, x_n) = \log_{x_1}((x_2)^2) + \cdots + \log_{x_{n-1}}((x_n)^n) + \log_{x_n} x_1.$$

Determine the minimum value of $f(x_1, \ldots, x_n)$ over S.

Solution. The answer is $n\sqrt[n]{n!}$. Note that

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= \log_{x_1}((x_2)^2) + \cdots + \log_{x_{n-1}}((x_n)^n) + \log_{x_n} x_1 \\ &= \left(\sum_{i=1}^{n-1} \log_{x_i} x_{i+1}^{i+1}\right) + \log_{x_n} x_1. \end{aligned}$$

First of all note that for $a, b \in (0, 1)$,

$$a < 1 \implies \log_b a > \log_b(1) = 0$$

therefore, every summand is positive and we can apply AM-GM inequality.

The following claim is crucial for applying AM-GM inequality later on.

Claim — $\left(\prod_{i=1}^{n-1} \log_{x_i} x_{i+1}^{i+1}\right) = \dfrac{n!}{\log_{x_n} x_1}$ for all $n \geq 2$.

Proof. The proof is by induction on n. For the base case of $n = 2$, observe that

$$\prod_{i=1}^{2-1} \log_{x_i} x_{i+1}^{i+1} = \log_{x_1} x_2^2 = 2! \cdot \log_{x_1} x_2 = \frac{2!}{\log_{x_2} x_1}$$

where the last equality is due a change of base formula to base x_1. This verifies the base case. For the **induction hypothesis**, assume that

$$\left(\prod_{i=1}^{m-1} \log_{x_i} x_{i+1}^{i+1}\right) = \frac{m!}{\log_{x_m} x_1}$$

for some $m \geq 2$. We need to show that holds true for $m+1$ as well. For the **induction step**, observe

$$\begin{aligned} \prod_{i=1}^{m} \log_{x_i} x_{i+1}^{i+1} &= \prod_{i=1}^{m-1} \log_{x_i} x_{i+1}^{i+1} \cdot \left(\log_{x_m} x_{m+1}^{m+1}\right) \\ &= \frac{m!}{\log_{x_m} x_1} \cdot (m+1) \cdot \log_{x_m} x_{m+1} \quad (\text{By } \textbf{induction hypothesis}) \\ &= (m+1)! \cdot \frac{\log_{x_m} x_{m+1}}{\log_{x_m} x_1} = (m+1)! \cdot \log_{x_1} x_{m+1} \end{aligned}$$

$$= \frac{(m+1)!}{\log_{x_{m+1}} x_1}$$

where we have continuously applied change of base formula of logarithms. This completes the induction step and the proof to the claim. □

¶ Proof of Lower Bound We first Simply applying AM-GM Inequality to the n summands yields

$$\begin{aligned}\frac{\left(\sum_{i=1}^{n-1} \log_{x_i} x_{i+1}^{i+1}\right) + \log_{x_n} x_1}{n} &\geq \sqrt[n]{\left(\prod_{i=1}^{n-1} \log_{x_i} x_{i+1}^{i+1}\right) \cdot \log_{x_n} x_1} \\ &= \sqrt[n]{\left(\frac{n!}{\log_{x_n} x_1}\right) \cdot \log_{x_n} x_1} \\ &= \sqrt[n]{n!}\end{aligned}$$

Multiplying both the sides by n gives us

$$f(x_1, x_2, \ldots, x_n) \geq n \cdot \sqrt[n]{n!}$$

which is what we wanted to show.

¶ Proof of Attainability of Bound We will show that the lower bound given above is achieved when

$$x_i = x_1^{\frac{n!}{i!t^{n-i+1}}} \iff \log_{x_1} x_i = \frac{n!}{i!t^{n-i+1}}$$

for $i = 1, 2, 3, \ldots, n$ where $t = \sqrt[n]{n!}$ where x_1 can be any real in $(0, 1)$. Also, note that since $\frac{n!}{i!t^{n-i+1}} > 0$, we know that

$$x_i = x_1^{\frac{n!}{i!t^{n-i+1}}} < x_1^0 = 1$$

which ensures $0 < x_i < 1$ for $i = 1, 2, 3, \ldots, n$.

We will look at every summand individually. First observe that

$$x_n = x_1^{1/t} \implies \log_{x_n} x_1 = t.$$

For any other summand $\log_{x_i} x_{i+1}^{i+1}$ where $1 \leq i \leq n-1$, observe that

$$\begin{aligned}\log_{x_i} x_{i+1}^{i+1} &= (i+1)\frac{\log_{x_1} x_{i+1}}{\log_{x_1} x_i} \\ &= (i+1) \cdot \frac{n!}{(i+1)!t^{n-i}} \cdot \frac{i!t^{n-i+1}}{n!} \\ &= t.\end{aligned}$$

So, finally

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n-1} \log_{x_i} x_{i+1}^{i+1} + \log_{x_n} x_1 = (n-1)t + t = nt = n\sqrt[n]{n!}$$

which **completes the solution.** ■

4. UGB 2026/4

Problem 4 (UGB 2026/4)

Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that

$$f(x+y) + f(x-y) = 2(f(x) + f(y)) \text{ for all } x, y \in \mathbb{R}.$$

a) If f' denotes the derivative of f, show that

$$f'(x+y) - f'(x-y) = 2f'(y) \text{ for all } x, y \in \mathbb{R}.$$

b) Further assume that $f''(0) = 2$, where f'' is the second derivative of f. Determine f.

Solution. Denote the functional equation $f(x+y) + f(x-y) = 2\left(f(x) + f(y)\right)$ by (♣). Observe that putting $x = y$ gives us $2f(0) = 4f(0) \implies f(0) = 0$. Putting $x = 0$ in (♣) yields

$$f(y) + f(-y) = 2f(y) \implies f(-y) = f(y) \text{ for all } y \in \mathbb{R}.$$

which proves that f is even. We will solve each part individually.

¶ Solution to part a) Fix $x = x_0 \in \mathbb{R}$ in (♣). Since f is given to be differentiable(twice differentiable in fact), so we can differentiate both sides of (♣) with respect to y. This yields

$$\begin{aligned}\frac{d}{dy}\Big(f(x_0+y) + f(x_0-y)\Big) &= \frac{d}{dy}\Big(2f(x_0) + 2f(y)\Big)\\ \implies f'(x_0+y) - f(x_0-y) &= 2f'(y) \text{ for all } y \in \mathbb{R}.\end{aligned}$$

Since x_0 was fixed arbitrarily, the above equation is also true for all $x \in \mathbb{R}$. Thus, we get

$$f'(x+y) - f'(x-y) = 2f'(y) \text{ for all } x, y \in \mathbb{R}. \qquad (♠)$$

This **finishes the proof of part a).** ■

¶ Solution to part b) We will prove that $f(x) = x^2$ is the only possible f satisfying the functional equation. As

$$f(x+y) + f(x-y) = (x+y)^2 + (x-y)^2 = 2(x^2+y^2) = 2\left(f(x) + f(y)\right)$$

the claimed characteriziation indeed works. We will now prove that this the only possible f satisfying (♣).

f is given to be twice-differentiable, so we will differentiate both sides of (♠) with respect to x this time. This involves first fixing y arbitrarily like we did in **part a)**. We'll skip these details this time.

$$\begin{aligned}\frac{d}{dx}\Big(f'(x+y) - f'(x-y)\Big) &= \frac{d}{dx}\Big(2f'(y)\Big)\\ \implies f''(x+y) - f''(x-y) &= 0\\ \implies f''(x+y) &= f''(x-y) \text{ for all } x, y \in \mathbb{R}.\end{aligned}$$

Replace x with $x + y$ in the above equation to get

$$f''(x + 2y) = f''(x) \text{ for all } x, y \in \mathbb{R}.$$

Finally, replace y by $y/2$ and then put $x = 0$, with the fact $f''(0) = 2$(given in the problem statement) which gives

$$f''(y) = f''(0) = 2 \text{ for all } y \in \mathbb{R}.$$

This proves that f'' identically 2 on $\mathbb{R}$. This is a very simple differential equation to solve, integrating both sides once gives

$$f'(x) = \int 2\, dx = 2x + C \text{ for come constant } C \in \mathbb{R}.$$

Integrating once more gives

$$f(x) = \int 2x + C\, dx = x^2 + Cx + D \text{ for some constants } C, D \in \mathbb{R}.$$

Since $f(0) = 0$, we get $D = 0$. Using $f(x) = f(-x)$, we get

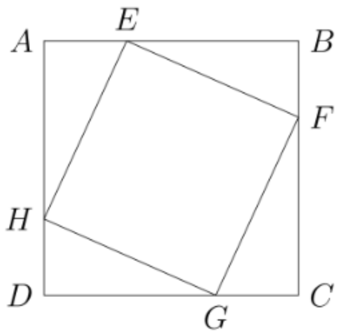
$$f(x) = x^2 + Cx = f(-x) = x^2 - Cx \implies 2Cx = 0 \text{ for all } x \in \mathbb{R}.$$

Since $2Cx = 0$ for any $x \in \mathbb{R}$, putting $x = 1$ forces $C = 0$ which gives $f(x) = x^2$ for all $x \in \mathbb{R}$, **completing the solution to part b)**. ■

5. UGB 2026/5

Problem 5 (UGB 2026/5)

In the figure below, $ABCD$ is a square. The points E, F, G, H lie on AB, BC, CD, DA, respectively, such that $EFGH$ is a square. The perimeter of $ABCD$ exceeds the perimeter of $EFGH$ by 32 units. Determine the length of the radius of the in-circle of ΔEBF.



Solution. The answer is 4 units. Let the side-length of square $ABCD$ be y and the side-length of square $EFGH$ be x. By the given perimeter condition,

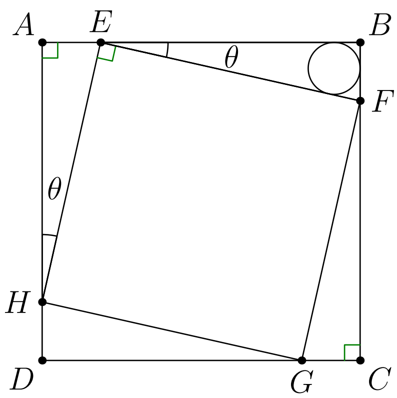
$$4y = 4x + 32 \implies y = x + 8.$$

We will now prove the following important claim.

Claim — $\triangle EFB \cong \triangle HEA \cong \triangle GHD \cong \triangle FGC$.

Proof. Let $\theta := \angle BEF$, then

$$\angle AEH = 180^\circ - (\angle HEA + \angle BEF) = 180^\circ - (90^\circ + \theta) = 90^\circ - \theta.$$



By angle sum in $\triangle AEH$, $\angle AHE = \theta$. Note that $\overline{HE} = \overline{EF}$(equal sides of square $EFGH$), $\angle HAE = \angle EBF = 90^\circ$ and $\angle BEF = \angle AHE = \theta$, so by AAS criteria $\triangle EFB \cong HEA$. The other congruencies can be proven in a similar fashion. This **proves our claim.** □

The corresponding parts of the congruent triangles yield $\overline{BF} = \overline{AE}$. From here, we will present two finishes to this problem.

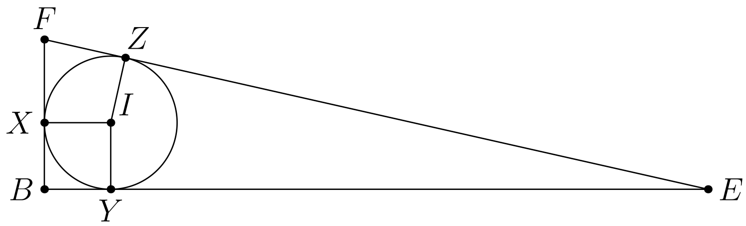
¶ First Finish, Direct Formula We prove the well-known theorem for inradius of a right triangle.

Theorem (Inradius of Right Triangle)
Denote the inradius of $\triangle EBF$ right-angled at B by r, then

$$r = \frac{f+e-b}{2}$$

where $f = \overline{EB}$, $e = \overline{BF}$ and $b = \overline{EF}$

Proof. If I is incenter of $\triangle EBF$, then let X, Y, Z be feet of perpendiculars from I to BF, EB and FE respectively.



Since $\angle IYB = \angle IXB = \angle XBY = 90°$, so we by angle sum of quadrilateral $IXBY$, we get $\angle XIY = 90°$ which tells us $IXBY$ is a rectangle. Moreover, $IXBY$ is a square since $\overline{IX} = \overline{IY}$. Finally,

$$b = \overline{FE} = \overline{EZ} + \overline{ZF} = \overline{EY} + \overline{FX} = (f-r) + (e-r) \implies r = \frac{f+e-b}{2}$$

where we used $\overline{EZ} = \overline{EY}$ and $\overline{FZ} = \overline{FX}$ since length of tangents drawn from an external point are equal The **proof to the theorem is now complete.** □

To conclude note that

$$r = \frac{f+e-b}{2} = \frac{\overline{BE}+\overline{BF}-\overline{FE}}{2} = \frac{\overline{BE}+\overline{AE}-x}{2} = \frac{\overline{AB}-x}{2} = \frac{x+8-x}{2} = 4$$

which **finishes the first solution.** ■

¶ Second Finish, Algebraic Retain all the notation from previous solution. We'll use the more standard formula,

$$r = \frac{\Delta}{s}$$

where Δ and s is the area and semi-perimeter of $\triangle EBF$. We know $\Delta = \frac{1}{2} \cdot ef$ and $s = \frac{1}{2} \cdot (e+b+f)$. Simplifying we get

$$r = \frac{fe}{f+e+b}$$

We know that

$$f+e+b = \overline{BE} + \overline{BF} + b = \overline{BE} + \overline{AE} + b = \overline{AB} + b = x+8+x = 2x+8.$$

By Pythagoras Theorem on $\triangle EBF$, we get $f^2 + e^2 = b^2 = x^2$. This gives

$$fe = \frac{(f+e)^2 - e^2 - f^2}{2} = \frac{(x+8)^2 - x^2}{2} = 8x + 32$$

Finally,

$$r = \frac{fe}{f+e+b} = \frac{8x+32}{2x+8} = \frac{8\cancel{(x+4)}}{2\cancel{(x+4)}} = 4$$

and **we're done.** ■

6. UGB 2026/6

Problem 6 (UGB 2026/6)

Let P be a polynomial of degree greater than or equal to one, with real coefficients, such that $P(x) \geq 0$ for all $x \in \mathbb{R}$.

a) Suppose $x_0 \in \mathbb{R}$ is such that $P(x_0) = 0$. Show that there exist a positive integer n and a polynomial S with real coefficients such that $S(x_0) \neq 0$ and

$$P(x) = (x - x_0)^{2n} S(x) \text{ for all } x \in \mathbb{R}.$$

b) Hence or otherwise, show that there exist polynomials Q and R with real coefficients such that $R(x) > 0$ for all $x \in \mathbb{R}$ and

$$P(x) = (Q(x))^2 R(x) \text{ for all } x \in \mathbb{R}.$$

Solution. We first prove a well-known preparatory lemma. A root α of P is said to be a repeated root of P if its multipilicity is at least 2.

Lemma (Repeated Roots)
Let α be a root of the polynomial P, then α is a repeated root of $P \iff P'(\alpha) = 0$.

Proof. Suppose first that α is a repeated root of P. Then

$$P(x) = (x - \alpha)^2 Q(x)$$

for some polynomial Q. Differentiating, we obtain

$$P'(x) = 2(x - \alpha)Q(x) + (x - \alpha)^2 Q'(x).$$

Evaluating at $x = \alpha$ yields $P'(\alpha) = 0$.

Conversely, suppose that $P'(\alpha) = 0$. Since α is a root of P, by the Factor Theorem, $P(x) = (x - \alpha)Q(x)$ for some polynomial Q. Differentiating gives

$$P'(x) = Q(x) + (x - \alpha)Q'(x).$$

Evaluating at $x = \alpha$, we find that $Q(\alpha) = P'(\alpha) = 0$. Applying the Factor Theorem once again, we conclude that $(x - \alpha)$ divides $Q(x)$. Hence

$$P(x) = (x - \alpha)^2 R(x)$$

for some polynomial R, showing that α is a repeated root of P. □

First of all observe that $\deg(P)$ must be even because all odd degree polynomials have range $\mathbb{R}$, thus we only need to prove the hypothesis for even degree polynomials P. We will solve both the parts with induction, but on different variables.

¶ Solution to part a) We prove the following claim.

Claim — x_0 is a repeated root of P.

Proof. Since x_0 is given to be a root of P, by the lemma proven above, it suffices to prove $P'(x_0) = 0$. For any $h > 0$, $P(x_0 + h) - P(x_0) = P(x_0 + h) \geq 0$, which means that

$$\text{RHD}_{x_0} = \lim_{h \to 0^+} \frac{P(x_0 + h) - P(x_0)}{h} \geq 0$$

where RHD_{x_0} means the right-hand derivative at x_0. For any $h > 0$, $P(x_0 - h) - P(x_0) = P(x_0 - h) \geq 0$. Now,

$$\text{LHD}_{x_0} = \lim_{h \to 0^+} \frac{P(x_0 - h) - P(x_0)}{-h} \leq 0$$

where LHD_{x_0} means the left-hand derivative at x_0.

Recall that polynomials are differentiable everywhere, so P must be differentiable at x_0. Also, P is differentiable at x_0 if and only if the $\text{LHD}_{x_0} = \text{RHD}_{x_0}$. This means

$$0 \leq \text{RHD}_{x_0} = \text{LHD}_{x_0} \leq 0 \implies \text{RHD}_{x_0} = \text{LHD}_{x_0} = 0 \implies P'(x_0) = 0$$

which proves our claim. □

We will now solve the problem by induction, by inducting on the degree of P.

Claim (Essentially part a) — Any non-negative even degree polynomial P if posesses a root x_0 must have an even multiplicity.

Proof. As advertised earlier, we will prove this by induction on $\deg(P)$. So let $\deg(P) = 2d$. We will first deal with the base of $d = 1$. Thus we need to show that if x_0 is the root of the quadratic $ax^2 + bx + c (a,\, b,\, c \in \mathbb{R})$, then α is a double root of P. By the *Factor Theorem*, we can write

$$ax^2 + bx + c = (x - x_0)(ux + v)$$

where u, v are some real constants. Any quadratic polynomial with two distinct roots **can NOT** be non-negative. This is easy to see, if $R(x) = r(x - \alpha)(x - \beta)$, then R either negative on the interval $\big(\min(\alpha, \beta), \max(\alpha, \beta)\big)$ (based upon the sign of r). Thus $ux + v$ must have x_0 as a root. This resolves the base case.

As an **induction hypothesis**, assume that all non-negative polynomials with degree $2d'$, if possess a root x_0, then multiplicity of x_0 is always even.

For the **induction step**, we wish to show that all non-negative polynomials of degree $2(d' + 1) = 2d' + 2$ with root x_0, posseses x_0 as a root with even multiplicity. So, let Q be an arbitrary non-negative polynomial with $\deg(Q') = 2d' + 2$ with root x_0. By the claim proven above, we know x_0 is a repeated root. Therefore, we can write

$$Q(x) = (x - x_0)^2 T(x)$$

where $\deg(T) = 2d'$. If x_0 is **NOT** a root of T, then we are done. If x_0 is a root of T, then by the **induction hypothesis**(the hypothesis applies since $\deg(T)$ is even and $T(x) = Q(x)/(x - x_0)^2 \geq 0$), T must possess x_0 as a root with an even multiplicity. So, we can write $T(x) = (x - x_0)^{2m} U(x)$ where $U(x_0) \neq 0$. This gives

$$Q(x) = (x - x_0)^{2m+2} S(x)$$

with $S(x_0) \neq 0$. This proves the claim. □

The above claim essentially **finishes the proof to part a).** ■

¶ **Solution to part b)** The proof is once again by induction, but we will use strong induction this time.

> **Claim** (Essentially part b) — All non-negative even degree polynomials P with $\deg(P) = 2d$ can be written as
> $$P(x) = Q(x)^2 \cdot R(x)$$
> with $R > 0$.

Proof. We will strong induct on d. The base case $d = 1$, wants us to show that any non-negative quadratic polynomial P can be written in the form $Q(x)^2 \cdot R(x)$. If P doesn't have have any root, then $P > 0$. Then we can pick $Q(x) = 1$ and $R = P$. If x_0 is a root of P, then by **part a)**, we know that x_0 is a double root. Therefore,

$$P(x) = a(x - x_0)^2$$

where $a > 0$. So, here picking $Q(x) = x - x_0$ and $R(x) = a$ works.

As an **induction hypothesis**, assume that all non-negative polynomials P of degree $2k$ where $k = 1, 2, \ldots, d'$ can be written in the form $Q(x)^2 R(x)$ where $R > 0$.

For the **induction step**, we need to show that any non-negative polynomial T with $\deg(P) = 2(d' + 1) = 2d' + 2$ can also be written in the form $Q(x)^2 R(x)$. If T doesn't have a root, then picking $Q(x) = 1$ and $R(x) = 1$ is enough.

If T has a root α, then by **part a)**,

$$T(x) = (x - x_0)^{2m} U(x)$$

with $U(x_0) \neq 0$. Clearly $\deg(U)$ is even and $U \geq 0$, so applying the **induction hypothesis** tells us that there exists polynomials $A, B \in \mathbb{R}[x]$ such that $U(x) = A(x)^2 \cdot B(x)$ with $B > 0$. This yields

$$T(x) = (x - x_0)^{2m} \cdot A(x)^2 \cdot B(x) = \left((x - x_0)^m \cdot A(x)\right)^2 \cdot B(x)$$

Picking $Q(x) = (x - x_0)^m \cdot A(x)$ and $R = B$ completes the strong induction. □

This **completes the solution to part b)** and the concludes the entire solution. ■

7. UGB 2026/7

Problem (UGB 2026/7)

Suppose P is a polynomial with integer coefficients.

a) Show that for distinct integers m and n, $\frac{P(m)-P(n)}{m-n}$ is an integer.

b) Hence or otherwise, prove that there do not exist distinct integers a, b, c such that
$$P(a) = b, \quad P(b) = c, \quad \text{and} \quad P(c) = a.$$

Solution. We will split the solution into two parts.

¶ Solution to part a) Write
$$P(x) = \sum_{i=0}^{d} a_i x^i$$
where $a_i \in \mathbb{Z}$ for $i = 0, 1, \ldots, d$. Since $a_i \cdot m^i \in \mathbb{Z}$ for any $m \in \mathbb{Z}$ and $m - n \in \mathbb{Z}$, the problem statement is equivalent to showing $m - n \mid P(m) - P(n)$. For distinct integers m, n
$$P(m) - P(n) = \sum_{i=0}^{d} a_i(m^i - n^i) = \sum_{i=1}^{d} a_i(m^i - n^i)$$
since $a_0(m^0 - n^0) = 0$. Recall the following standard lemma.

Lemma (Standard Divisibility)
Given $m, n \in \mathbb{Z}$, $m - n \mid m^k - n^k$ for all $k \in \mathbb{Z}_{\geq 1}$.

Proof. We shall strong induct on k. The base case of $k = 1$ is trivial. As an **induction hypothesis**, assume that $m - n \mid m^t - n^t$ for $t = 1, 2, \ldots, k$ for $k \geq 1$. For the **induction step**, we need to show $m - n \mid m^{k+1} - n^{k+1}$. By the **induction hypothesis**, we know that
$$m - n \mid m^k - n^k \implies m - n \mid m(m^k - n^k) = m^{k+1} - mn^k$$
$$m - n \mid m^k - n^k \implies m - n \mid m(m^k - n^k) = nm^k - n^{k+1}$$
Adding both the division relations we get
$$m - n \mid m^{k+1} - n^{k+1} - mn(m^{k-1} - n^{k-1})$$
By our *strong* **induction hypothesis**, $m - n \mid m^{k-1} - n^{k-1}$, which ultimately gives
$$m - n \mid m^{k+1} - n^{k+1}$$
completing the **induction step** and the proof of the lemma. □

Due to the above lemma, we can now write $m^k - n^k = (m-n)q_k$ where $q_k \in \mathbb{Z}$ and q_k is dependent upon k where $k \geq 1$. Thus,
$$P(m) - P(n) = \sum_{i=1}^{d} a_i(m^i - n^i) = \sum_{i=1}^{d} a_i \cdot q_i(m-n) = (m-n) \cdot \left(\sum_{i=1}^{d} a_i q_i\right)$$
Since a_i, q_i are integers for $i = 1, 2, \ldots, d$, $\sum a_i q_i \in \mathbb{Z}$ as well. This gives $m - n \mid P(m) - P(n)$ which finishes the **proof to part a).** ■

¶ Solution to part b) Assume for the sake of contradiction that there exists distinct a, b and c satisfying

$$P(a) = b, \quad P(b) = c, \quad \text{and} \quad P(c) = a.$$

With the help of **part a)**, we now know that

$$a - b \mid P(a) - P(b) = b - c.$$

Similarly with $b - c \mid P(b) - P(c)$ and $c - a \mid P(c) - P(a)$, we will get the following division relations. This gives

$$\begin{aligned} a - b &\mid b - c \\ b - c &\mid c - a \\ c - a &\mid a - b \end{aligned}$$

We now prove a short standard lemma.

Lemma (Divisibility Inequality)
For $x, y \in \mathbb{Z}$ where $x, y \neq 0$ and $x \mid y$ then

$$|y| \geq |x|.$$

Proof. Since $x \mid y$, there exists some $k \in \mathbb{Z}$ such that $kx = y$. If $k = 0$, then $y = 0$ contradiction. Thus $|k| \geq 1$ and finally

$$1 \leq |k| = \frac{|y|}{|x|} \implies |x| \leq |y|.$$ □

Clearly, none of $a - b, b - c, c - a$ none of them are 0, otherwise we get contradiction on distinct nature of a, b, c. Note that

$$a - b \mid b - c \mid c - a \mid a - b.$$

Applying the lemma, we get

$$|a - b| \leq |b - c| \leq |c - a| \leq |a - b| \implies |a - b| = |b - c| = |c - a|.$$

We have $|a - b| = |b - c| \implies a - b = \pm(b - c)$. If $a - b = -(b - c)$, then $a = c$, a contradiction since a, b and c are distinct. Therefore, $a - b = b - c \implies 2b = a + c$. Similar reasoning on $|a - b| = |c - a|$ and $|b - c| = |c - a|$ gives us the following system.

$$\begin{aligned} 2c &= a + b \\ 2a &= b + c \\ 2b &= a + c \end{aligned}$$

Subtract the first two equations to get $2c - 2a = a - c \implies 3c = 3a \implies c = a$ which is again a contradiction. This **completes the the solution to part b).** ■

8. UGB 2026/8

Problem (UGB 2026/8)

Let k be a non-negative integer.

1. Show that there exists a unique polynomial P_k of degree $k+1$ with real coefficients such that
$$\sum_{i=1}^{n} i^k = P_k(n) \text{ for all } n \geq 1.$$
2. Determine the coefficient of x^{k+1} in $P_k(x)$.
3. Hence or otherwise, determine the coefficient of x^k in $P_k(x)$.

Solution. To be added...For now, the informal solution is available on the blog

https://satisfiedmagma.github.io/writings/posts/isi_2026_discussion/#-problem-8

Thank you! ■

9. Concluding Remarks

I pretty much have summed up everything on the blog. Although, I would like to mention that maybe the problems were on the easier side and a skilled candidate would have a really easy time, but once again the entrance paper has a lot of room to make error even in the simpler problems. Proof-writing is going to make the major difference for the people who reach the Merit List!

I hope you enjoyed reading the solutions and learnt something new from them. It was my very first blog on my website and I've tried my best. This is Magma, signing off ;)